Lesson 14: Implicit Differentiation

Warmup 1. $\frac{d}{dx}[x^2+\sin^2(x)] = 2x + 2(\sin x)\cdot\frac{d}{dx}[\sin x]$

$= \boxed{2x + 2\sin x\cos x}$

Chain Rule
OUT $= u^2$
IN $= \sin x$

2. Find $\frac{dy}{dx}$ if $y = 3x$.

$\frac{dy}{dx} = 3\cancel{\frac{dx}{dx}}^{1}$

Ex 1 Find $\frac{d}{dx}[x^2 + f(x)^2] = 2x + 2(f(x))\cdot\frac{d}{dx}[f(x)]$

$= 2x + 2f(x)f'(x)$

Chain Rule
OUT $= u^2$
IN $= f(x)$

Ex 2 Find $\frac{d}{dx}[x^2+y^2]$ if y is a function of x.

Chain Rule
OUT $= u^2$
IN $= y$ $\left(\frac{d}{dx}[y] = \frac{dy}{dx} \text{ or } y'\right)$

$\frac{d}{dx}[x^2+y^2] = 2x + 2y\cdot\frac{dy}{dx}$

Def An implicit function is a function (or equation) where the dependent variable (y, usually) is not isolated (no $y = f(x)$)

Examples $x^2+y^2=9 \leftarrow$ circle w/ $r=3$, center $(0,0)$

$xy = 72y^2$

$y-1 = x+3$

When we have an implicit function, we use implicit differentiation. to find $\frac{dy}{dx}$.

① Take the derivative of both sides with respect to x. When taking the derivative of a function of y, multiply by $\frac{dy}{dx}$. (see Ex 2).

② Solve for $\frac{dy}{dx}$.

Ex 3 Find $\frac{dy}{dx}$ if $x^2+y^2=1$.

① $\frac{d}{dx}[x^2+y^2]=\frac{d}{dx}[1]$

$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]=0$

$2x+2y\cdot\frac{dy}{dx}=0$

② $-2x \qquad -2x$

$2y\frac{dy}{dx}=-2x$

$\frac{dy}{dx}=\frac{-\cancel{2}x}{\cancel{2}y}=\boxed{\frac{-x}{y}}$

Now find the tangent line at $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad (x_0,y_0)$

$\frac{dy}{dx}=\frac{-\sqrt{3}/2}{1/2}\cdot\frac{2}{2}$

$=-\sqrt{3}$

$\boxed{y-\frac{1}{2}=-\sqrt{3}\left(x-\frac{\sqrt{3}}{2}\right)}$

Ex 4 Find $\frac{dy}{dx}$ at $(2,1)$ if $xy^2=y+1$.

① $\frac{d}{dx}[xy^2]=\frac{d}{dx}[y+1]$

Product $\rightarrow$ $\frac{d}{dx}[x]\,y^2+x\frac{d}{dx}[y^2]=1\cdot\frac{dy}{dx}+0$

$1\cdot y^2+x\cdot 2y\cdot\frac{dy}{dx}=\frac{dy}{dx}$

at $(2,1)$: $1+2\cdot 2\frac{dy}{dx}=\frac{dy}{dx}$

$1+4\frac{dy}{dx}=\frac{dy}{dx} \quad \leftarrow -\frac{dy}{dx}$

$1+3\frac{dy}{dx}=0$

$\boxed{\frac{dy}{dx}=-\frac{1}{3}}$

Ex 5 $\quad \frac{1}{y} - \cos^2 x = 0$

① $\quad y^{-1} - (\cos x)^2 = 0$

$$\frac{d}{dx}\left[y^{-1} - (\cos x)^2\right] = \frac{d}{dx}[0]$$

$$-y^{-2} \cdot \frac{dy}{dx} - 2\cos x(-\sin x) = 0$$

↑
Chain Rule!

$$-y^{-2}\frac{dy}{dx} + 2\sin x\cos x = 0$$

② $\quad (-y^2)\ -y^{-2}\frac{dy}{dx} = -2\sin x\cos x\ (-y^2)$

$$\boxed{\frac{dy}{dx} = -2y^2\sin x\cos x}$$

Ex 6 $\quad \ln(x^2+xy) = y^{1/2}$

① $\quad \frac{d}{dx}\left[\ln(x^2+xy)\right] = \frac{d}{dx}\left[y^{1/2}\right]$

Chain Rule
OUT $= \ln u$
IN $= x^2+xy$

$$\frac{1}{x^2+xy} \cdot \frac{d}{dx}\left[x^2+xy\right] = \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx}$$

$$\frac{1}{x^2+xy} \cdot \Big(2x + \underbrace{1 \cdot y + x \cdot \frac{dy}{dx}}_{\text{Product Rule}}\Big) = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

② Solve for $\frac{dy}{dx}$.

Ex 7 $\quad e^{x/y} = 1$

$$\frac{d}{dx}\left[e^{x/y}\right] = \frac{d}{dx}[1]$$

Chain Rule
OUT $= e^u$
IN $= \frac{x}{y}$

$$e^{x/y} \cdot \frac{d}{dx}\left[xy^{-1}\right] = 0$$

$$e^{x/y}\left(1 \cdot y^{-1} + x\frac{d}{dx}\left[y^{-1}\right]\right) = 0$$

$$e^{x/y}\left(y^{-1} + x\left(-y^{-2} \cdot \frac{dy}{dx}\right)\right) = 0$$

② Solve for $\frac{dy}{dx}$.

Ex 8 Find $\frac{d}{dt}[\sin x + xy]$ if x and y are functions of t.

$= (\cos x)\frac{dx}{dt} + \frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}$.